3.12 *Math Focus.* Math Trick #13: The Law of Conservation of Energy

In Section 2.13, we have already seen a useful rule called conservation of momentum. We can write down another useful conservation rule by defining something called *energy*. Energy is a very powerful concept, and you have probably heard the word used many times in discussion of things like the "energy crisis" or "saving energy," etc. It is very important to remember, however, that the word energy has a very specific meaning, which we will define here. Sometimes New Age religions use scientific words like energy to refer to vague ideas they have, without using the proper scientific meaning. Sometimes we also use the word to refer to our feelings, such as "I don't have any energy today." That is okay, but don't mix up those meanings with the scientific meaning!

We write the *kinetic energy* as E, and define it as

$$E = \frac{1}{2}mv^2,$$

where m is the mass of some object and v is its speed. From this definition, we can see that the unit of energy is the funny combination, kg-m^2/s^2. This "funny" combination of units is commonly called a *Joule* (after the scientist, Joule), and is written

$$1\ \text{J} = \text{N-m} = \frac{\text{kg-m}^2}{s^2}.$$

Suppose we want to find the change in kinetic energy of an object over a very small time interval dt. Over this interval, the velocity of the object will change a small amount, from $v_1 = v$ to $v_2 = v + dv$. Then

$$\begin{aligned} dE = E_2 - E_1 &= \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2 \\ &= \frac{1}{2}m\left[(v+dv)^2 - v^2\right] \\ &= \frac{1}{2}m\left[v^2 + 2v(dv) + (dv)^2 - v^2\right] \\ &= \frac{1}{2}m(2v\ dv). \end{aligned}$$

Here we have dropped the term $(dv)^2$ because it is small compared to $v(dv)$, since we are concerned only with the velocity a short time later, so that dv is small. Continuing on,

$$\begin{aligned} dE &= m\left(\frac{dx}{dt}\right)dv \\ &= m\left(\frac{dv}{dt}\right)dx \\ &= ma\ dx. \end{aligned}$$

Here we have used the definition of the acceleration $a = dv/dt$. Next we recall that $F = ma$, which means

$$\begin{aligned} dE &= F dx \\ &= -dU. \end{aligned}$$

We have defined a new quantity, U, called the *potential energy*, such that $dU = -F dx$. It may seem strange that we have put in a minus sign in our definition, but this is convenient if we rewrite the above equation as

$$dE + dU = d(E + U) = 0. \tag{3.4}$$

According to the above calculation, then, Newton's second law, $F = ma$, implies that the total of the kinetic energy and the potential energy does not change, that is, the total energy is *conserved*, just as we found for the total *momentum* previously.

What this means is that energy can be traded off between one type or another, but never destroyed. For example, a rock sitting on a ledge has potential energy. If it is knocked off, it will lose potential energy (it gets lower), but it will gain kinetic energy (speed). Gravity is not the only force which creates potential energy. Other forces, such as electric force, which we will study later, also create potential energy.

If we assume that the force of gravity is nearly constant near the surface of the earth, then the potential energy of an object with mass m_1 is just

$$U = -Fx = m_1 a_g x, \tag{3.5}$$

where a_g is the acceleration due to gravity at the earth's surface, found in Section 3.5, and x is the height. If we take upward as positive, then since the force of gravity points downward (so that we must give it a negative sign), the potential energy of an object gets greater the higher it is, since the two negatives signs cancel. Note that we are free to define the place where $U = 0$ anywhere we want, since according to (3.4) we only need to keep track of the *change* in U. Therefore, in the definition $U = -Fx$, we can pick any convenient place for $x = 0$. When we are talking about gravity, it will often be convenient to define the surface of the earth as $x = 0$.

Energy conservation can help us to solve some problems very easily. For example, suppose you shoot a bullet into the air at speed v. How fast will the bullet be going when it hits the ground?

When the bullet is shot into the air, as it goes up it will gain potential energy. This means it will slow down (lose kinetic energy). Eventually, it will turn around and come back down. When it gets back to the earth, it will have lost all the potential energy it gained. Therefore, it must have gained back all the kinetic energy it had (as long as it did not lose much energy to some other place, such as wind friction). That means it must have exactly the same speed when it comes down as it had when it left the gun. No wonder firing a gun into the air is dangerous!

We can also use this law quantitatively. For example, suppose a cannon ball is fired straight up into the air at a speed of 100 m/s. How far up does it go? To find this, we use

Equation (3.5) above. We know that at the top of its travel, all of its kinetic energy will be converted into potential energy. Therefore at the top, $U = E = \frac{1}{2}mv^2$. We can solve this equation for x, as follows:

$$\begin{aligned} m a_g x &= \frac{1}{2} m v^2 \\ x &= \frac{mv^2}{2ma_g} = \frac{v^2}{2a_g} = \frac{(100\ \mathrm{m/s})^2}{2(9.8\ \mathrm{m/s^2})} = 510\ \mathrm{m}. \end{aligned}$$

Notice that once again, the mass cancels out. The height depends only on the initial velocity.

Assignment:

1. According to Equation (3.5), energy should have the units of force times distance, or N-m. Show that this is the same as the definition of the Joule unit, using the definition of the Newton unit of force given in Section 2.10.

2. A typical bullet velocity from a gun is 500 m/s. How high will a bullet go if you shoot it into the air? Is it reasonable to try to shoot an airplane with a handgun? Of course, at its peak the bullet is not moving at all. How high will the bullet be when it is moving at 100 m/s? Remember that the potential energy gained at that point will be equal to the amount of kinetic energy *lost*.

3. If you can jump one foot high on the earth, how high could you jump on the moon? You do not need to use a lot of numbers. You should be able to make a simple argument using conservation of energy and proportionalities. Assume that your muscles would act the same (give you the same initial speed) on the moon as on the earth.